Architectonics of *Cleanness* Revisited

Ron Catterall

Abstract

Crawford has proposed that there is a complex geometrical structure in the poem *Cleanness* which is determined by the location of twelve of the thirteen large decorated capitals in the Nero A.x manuscript, together with the length of the poem. In support of this contention she noted eleven equalities and symmetries in the relations between the line positions of the capitals, and claimed that these relations were unique and would be destroyed if a capital was moved by a single line. Therefore she argued one has to accept that the capitals were placed deliberately and did not happen by chance. The geometrical basis for the structure was based on an identification of ratios of separations between the decorated capitals, with the Golden Ratio (0.618034), and with the diagonal of the unit square (1.414214).

If this was indeed a structure imposed by the poet, then one would expect to see the structure reflected in the subject matter. However, as others have noted, the locations of the decorated capitals do not correlate at all convincingly with the logical development of the material in the text.

The conjecture of deliberate placement rests upon the claim of uniqueness of the positioning of the decorated capitals. We have re-examined the proposal and find that the five of the eleven points noted by Crawford are redundant; they are logically equivalent to combinations of the remaining six. We also find that there are very many equivalent sets of capital placements determined by a set of five arbitrary constants, which would also satisfy all the equalities, symmetries, and ratios noted by Crawford. Taking into account the lack of significant contextual support for the placement in the Nero A.x manuscript, we conclude that the capitals do not define a deliberate and complex structure, and that there is a very reasonable probability (about 1 in 8) that they could have arisen purely by chance.

Table of Contents

1. Preface

There is an apparently complex structure of the poem *Cleanness* arising from the location of the large decorated capitals in the Nero A.x manuscript. Eleven equalities and symmetries in the relation between the line positions of the capitals were identified by Crawford [CRAWFORD93]. It was claimed that these relations would be destroyed if a capital was moved by a single line, and therefore one has to accept that the capitals were placed deliberately, and that their placement did not happen by chance. Taken together with the approximate equivalence between ratios of separations between the decorated capitals and the Golden Ratio and the diagonal of a square, it appears that the poem exhibits a very precise, complex, and geometrical structure for the poem.

Significant narrative and thematic structure in *Cleanness* has been proposed, but there is little or no correlation between this textual structure and that indicated by the decorated capitals, and one is left with two alternative explanations for this discrepancy, neither of which is particularly acceptable.

- The capitals were placed (by the scribe presumably) completely at random or at best with only a minimal study of the text.
- The poet produced a very complex frame for his new work and then virtually ignored it when he came to compose the poem.

This paper attempts to resolve this issue.

2. Introduction

There are thirteen large, decorated capitals in *Cleanness*, and Ms Crawford has demonstrated that there are eleven repetitions and symmetries between twelve of them and the end of the poem. Two of these relations required the addition of a final, virtual capital after the end of the poem at the non-existent line 1813. Crawford insisted that “*such a set of relationships could not happen by chance*” and that “*these repetitions represent a unique structure which could not occurred if the position of one of the decorated capital letters had been changed even by just one line*”, and indeed “*any change in the location of the decorated capitals marking the intervals would spoil their precision*”. Finally Ms Crawford claims that “*the decorated capital letters, as well as the final line count of the poem, are involved in the production of repeating intervals, a circumstance suggesting that the manuscript does record an intentional plan*”. The importance of the relative locations of these decorated capitals has been generally accepted, and Edwards [EDWARDS97] notes that “*Donna Crawford, in a subtle and important study, has seen them as exemplifying the poem’s ‘geometrical art’ because of the various ratios that are revealed through the intervals between them*”. Ms Crawford’s paper and its reception has left us with the impression that the location of the capitals is deliberate, and that they mark out a complex and unique structure in the poem.

Unfortunately there is little evidence in the text of the poem that the placement of the decorated capitals corresponds to significant textual structure. Gollancz [GOLLANCZ21], Menner [MENNER20] used the decorated capitals to subdivide the poem into thirteen sections for the purpose of presentation. Laurita Hill [HILL46] pointed out that “*four initials (l.345, l.601, l.781, l.1357) come at what corresponds to the beginning of chapters in the Vulgate; three initials (l.485, l.689, l.893) occur at sections which contain elaborations and other non-Biblical material; three come after shifts in Biblical story (l.193, l.249, l.1157); and one marks a transition from the story of Noah to the story of Abraham (l.557)*”.

Spendal [SPENDAL76] suggested that the decorated capitals mark off thematic rather than narrative sections and was able to find plausible thematic matches for all the capitals. My worry is that one can find other equally significant thematic breaks which are not enhanced by large decorated capitals.

There have been two more systematic and extended studies of the narrative structure which do not fit the locations of the decorated capitals too well. Spearing [SPEARING70] noted significant primary narrative breaks at lines 205, 545, 601, 1049, 1149, and 1805 with secondary breaks at 49, 177, 235, 249, 781, 1069, 1109, and 1333. This matches one primary and two secondary breaks out of fourteen; not a particularly impressive success rate. More recently, in their definitive rendering of the poems of the Nero A.x manuscript, Andrew and Waldron [ANDREW02] find primary narrative structure starting at lines 193, 545, 601, 1049, 1149, and 1805 with secondary structure starting at lines 23, 49, 235, 249, 677, 890, 973, 1001, 1157, 1333, 1651, and 1709 (1710-1804 is not covered). Two primary and two secondary hits out of 18. There is agreement between all of them that there is a primary break in the narrative at line 601, and a secondary break at line 249, but this is hardly supportive of the complex structure proposed by Ms Crawford.

Spearing [SPEARING70] (p.43, footnote 2) perhaps summed up the general thought: "*I think that the scribe used decorated initials with intelligence, but somewhat freely and without making a very close study of the parts into which the poem falls*". Given this admittedly very loose correlation between the positions of the decorated capitals and the content of the poem, how do we reconcile it with the very precise structure proposed by Crawford. Without any doubt the location of the decorated capitals is primary evidence; they are there, and any repetitions and symmetries between them are present in the Nero A.x manuscript. But, do they define a unique and deliberate structure imposed by the poet, or could they be just one of many different equivalent patterns: are there other combinations of line positions which yield the same relations.

All the relationships noted above are dependent on line numbers (integers) alone and by themselves do not support the geometrically inspired structure proposed by Ms Crawford. The argument for the proposed geometric structure rests upon rather more uncertain evidence, that ratios of differences between positions of decorated capitals are close to values of geometric significance. Unfortunately these "values of geometric significance" are not simple integers like line numbers, they are not only fractional, but they are also be irrational, that is, they cannot be represented by ratios of integers. We leave discussion of this secondary evidence to a later section and concentrate firstly upon the integral line positions as primary evidence.

We can question the insistence that the placement of these capitals constitutes a unique structure that would be destroyed by moving even one capital by one line position. Is there really a need for the capitals to be so exactly placed to generate the repetition and symmetries of the spaces between them. If the location of the eleven capitals in *Cleanness* were totally independent and the only and unique way to generate the repetitions and symmetries,then we would have to conclude along with Ms Crawford that the locations were chosen deliberately and could not have arisen by a chance occurrence.[i] Furthermore, the eleven criteria cited by Ms Crawford should be sufficient to uniquely define a set of eleven positions of the capitals: in mathematical terminology a set of eleven linear equations in eleven variables has a single unique solution. The critical proviso here is that the equations must be linearly independent - i.e. none of them can be derived from combinations of the others. In this paper we demonstrate that the eleven criteria are not linearly independent, and that very many other locations of the capitals exhibit exactly the same repetitions and symmetries, and, finally, that the complex structure proposed could well have arisen by chance.

[i] If we were to allow the line positions of eleven internal capitals to change by plus or minus one, then the probability of achieving the positions observed is 0.000005645 or about 177471 to 1 against them being positioned as they are by chance.

3. Investigation

We start by replacing actual line numbers of the decorated capitals by symbols, indicating that they are variable. As an example, we replace the line position of the sixth capital by the symbol "f", so that we can investigate the possibility of displacing that capital by ±1, ±2 etc. line positions on the relations noted by Ms Crawford. The positions of all the thirteen capitals and the end of the poem are now represented by the symbols a, b, … n.

a	**b**	**c**	**d**	**e**	**f**	**g**	**h**	**i**	**j**	**k**	**l**	**m**	**n**
1	125	193	249	345	485	557	601	689	781	893	1157	1357	1813

Subject to the condition:

$$a < b < c < d < e < f < g < h < i < j < k < l < m < n$$

Of the 14 possible locations, 2 are effectively fixed, the start and end of the poem at 1 and 1813 respectively, and since Crawford recorded nothing about the capital at line 1357, we set a=1, m=1357,[ii] and n=1813, so we are left with 11 "variables" (b, c, …l) which are all positive, integral, non-zero and in ascending order. The basic 9 repetitions recorded by Crawford are concerned with differences between these values. For reference and completion the full table of differences is presented in Table 1, "Difference Table".

[ii]The inclusion of the capital at line 1357 in the treatment is discussed in a later section Appendix B, *The Decorated Capital at Line 1357*.

Table 1. Difference Table

(b-a)												
124												
(c-a)	(c-b)											
192	68											
(d-a)	(d-b)	(d-c)										
248	124	56										
(e-a)	(e-b)	(e-c)	(e-d)									
344	220	152	96									
(f-a)	(f-b)	(f-c)	(f-d)	(f-e)								
484	360	292	236	140								
(g-a)	(g-b)	(g-c)	(g-d)	(g-e)	(g-f)							
556	432	364	308	212	72							
(h-a)	(h-b)	(h-c)	(h-d)	(h-e)	(h-f)	(h-g)						
600	476	408	352	256	116	44						
(i-a)	(i-b)	(i-c)	(i-d)	(i-e)	(i-f)	(i-g)	(i-h)					
688	564	496	440	344	204	132	88					
(j-a)	(j-b)	(j-c)	(j-d)	(j-e)	(j-f)	(j-g)	(j-h)	(j-i)				
780	656	588	532	436	296	224	180	92				
(k-a)	(k-b)	(k-c)	(k-d)	(k-e)	(k-f)	(k-g)	(k-h)	(k-i)	(k-j)			
892	768	700	644	548	408	336	292	204	112			
(l-a)	(l-b)	(l-c)	(l-d)	(l-e)	(l-f)	(l-g)	(l-h)	(l-i)	(l-j)	(l-k)		
1156	1032	964	908	812	672	600	556	468	376	264		
(m-a)	(m-b)	(m-c)	(m-d)	(m-e)	(m-f)	(m-g)	(m-h)	(m-i)	(m-j)	(m-k)	(m-l)	
1356	1232	1164	1108	1012	872	800	756	668	576	464	200	
(n-a)	(n-b)	(n-c)	(n-d)	(n-e)	(n-f)	(n-g)	(n-h)	(n-i)	(n-j)	(n-k)	(n-l)	(n-m)
1812	1688	1620	1564	1468	1328	1256	1212	1124	1032	920	656	456

The nine significant equalities of differences noted by Crawford, which imply the nested symmetries she noted, can be presented in symbolic form.

(345-1)=(689-345)=344	e-a=i-e	⇒	-2*e+i=-a	..........(1)
(125-1)=(249-125)=124	b-a=d-b	⇒	-2*b+d=-a	..........(2)
(689-485)=(893-689)=204	i-f=k-i	⇒	f-2*i+k = 0	..........(3)
(601-1)=(1157-557)=600	h-a=l-g	⇒	-g-h+l=-a	..........(4)
(557-1)=(1157-601)=556	g-a=l-h	⇒	-g-h+l=-a	..........(5)
(1157-125)=(1813-781)=1032	l-b=n-j	⇒	b-j-l=-n	..........(6)
(781-125)=(1813-1157)=656	j-b=n-l	⇒	b-j-l=-n	..........(7)
(601-193)=(893-485)=408	h-c=k-f	⇒	c-f-h+k = 0	..........(8)
(485-193)=(893-601)=292)	f-c=k-h	⇒	c-f-h+k = 0	..........(9)

Simple manipulation of the symbols as above shows that three of them, (5), (7), and (9), are identical with (4), (6), and (8), and thus there are only six independent relations between the symbols: (1), (2), (3), (4), (6), and (8).

In addition to the nine equalities listed above Crawford also noted that there were two instances where one difference was a simple multiple of another. First of all she noted that (h-c) (=408) is exactly twice (i-f) (=204), and also that (k-i) (=204, equation 3) is adjacent to (i-f). Symbolically this can be represented by

(i-f)+(k-i)=(h-c) ⇒ c-f-h+k = 0 ....(10)

Clearly the condition (10) is identical to (8) and represents no additional information.

Crawford also noted that (l-b)=(n-j)=1032 is exactly three times (e-a)=(i-e)=344. Again we express this symbolically.

3*(e-a)=(l-b)	⇒	-b-3*e+l=-3*a	..........(11)
3*(e-a)=(n-j)	⇒	+3*e+j=n+3*a	..........(12)
3*(i-e)=(l-b)	⇒	-b+3*e-3*i+l=0	..........(13)
3*(i-e)=(n-j)	⇒	-3*e+3*i+j=n	..........(14)
add 11 and 13	⇒	-2*b-3*i+2*l=-3*a	..........(15)
add 12 and 14	⇒	-3*i-2*j=-2*n-3*a	..........(16)
sub 16 from 15	⇒	-2*b+2*j+2*l=2*n	..........(17)

Relations (12) and (14) do not add any information, the identity (n-j)=(l-b) is given in (6). We now also see that (17) is identical with (6) multiplied by -2. Again there is no additional information and we are still left with six linear equations which are linearly independent in eleven variables, a clear indication that multiple solutions are possible, i.e. there are likely to be many combinations of b, c, … l that satisfy all the eleven criteria recorded by Crawford.

We now investigate these other possible combinations of positions of the decorated capitals. First a unique solution would require a set of eleven equations which would be expressed by the matrix equation

$$A \; . \; x \; = \; C$$

Where A is a matrix of eleven columns of the eleven coefficients of b, c, … l in eleven equations represented by the eleven rows, and x is a column vector of solutions with eleven rows corresponding to the values of the variables b, c, … l to be determined. C is the vector of eleven constants formed by re-arranging the equations to put the constants on the right hand side. In this case a set of unique line positions can be found from the product of the inverse of the matrix A (A^{-1}) and the constant vector, C.

$$x \; = \; A^{-1} \; . \; C$$

The claim that the location of the capitals in *Cleanness* is unique rests upon a supposition that the eleven relations noted by Ms Crawford are linearly independent. Unfortunately we have just demonstrated that these eleven relations are not linearly independent: there are only six independent relations, the other five can be inferred logically from the six. Although we cannot proceed by simple inversion of the matrix A, we can proceed to define a set of solutions, x. First we re-write equations (1), (2), (3), (4), (6), and (8) to replace the constants a and n by their values.

-2*e+i = -1	..........(1')
-2*b+d = -1	..........(2')
f-2*i+k = 0	..........(3')
-g-h+l = -1	..........(4')
b-j-l = -1813	..........(6')
c-f-h+k = 0	..........(8')

Thus the matrix A is

Table 2. Coefficient matrix - 6 equations 11 variables

$$A = \begin{vmatrix} 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \end{vmatrix}$$

and the constant vector C is

Table 3. Vector of constants

$$C = \begin{vmatrix} -1 \\ -1 \\ 0 \\ -1 \\ -1813 \\ 0 \end{vmatrix}$$

This is the matrix representation of an under-determined set of six consistent linear equations in M=11 unknowns. When the rank, R, of the matrix A is less than eleven, there is at least one solution vector (X) together with a set of M-R linearly independent vectors U_1, U_2, … U_{M-R}, and the general solution is given by the vector X plus any linear combination of all the vectors U. There are no other solutions.

Firstly, using standard techniques (described with worked examples by Pennington [PENNINGTON65]) we note that the rank of A is 6 and we solve the matrix equation to obtain X and a set of five U vectors. Now we can write the general solution as

Table 4. General Solution fitting all 11 Criteria, 6 equations

b		0.5		0		-0.5		0		0		0
c		0		-1		0		-1		1		0
d		0		0		-1		0		0		0
e		0.5		0		0		-0.25		-0.25		0
f		0		0		0		-1		0		0
g	=	1	$+C_1*$	1	$+C_2*$	0	$+C_3*$	0	$+C_4*$	0	$+C_5*$	-1
h		0		-1		0		0		0		0
i		0		0		0		-0.5		-0.5		0
j		1813.5		0		-0.5		0		0		1
k		0		0		0		0		-1		0
l		0		0		0		0		0		-1

As a simple illustration of this general solution we set C_1=-601, C_2=-249, C_3=-485, C_4=-893, C_5=-1157, we obtain the complete solution as observed in *Cleanness*: b=125, c=193, d=249, e=345, f=485, g=557, h=601, i=689, j=781, k=893, and l=1157. We only need to specify five constants to determine not only the positions of all eleven decorated capitals in *Cleanness* but the observed multiples as well.

In the present situation we need to restrict our interest to those solutions which generate a solution set of integral values: decorated capitals only occur at the start of a line; so we need to show that such integral solutions do exist.[iii] Note that the five C_i constants are in fact the negative of line po-

[iii]Many non-integral solutions do exist which also satisfy all the eleven criteria listed by Crawford. For example:

sitions for 5 of the decorated capitals, but the ordering does not have to be increasing. The complexity of the eleven relations among eleven capital locations noted by Crawford has been reduced to a choice of only five line numbers: all the others fall into place automatically. In Section 4, “*Results*” we explore the range of acceptable sets of line positions for decorated capitals which meet all the criteria noted by Crawford.

4. Results

Clearly there are an infinite number of possible solutions, all conforming to the scheme of capitals in *Cleanness* because the choice of values for $C_1 \ldots C_5$ is unlimited. However, many of these solutions, although conforming to all eleven constraints, are unacceptable. For example, setting C_1=-0.1, C_2=-0.2, C_3=-0.3, C_4=-0.4, and C_5=+0.5 yield the sequence b=0.6, c=0, d=0.2, e=0.675, f=0.3, g=0.4, h=0.1, i=0.35, j=1814.1, k=0.4, and l=-0.5 which is clearly inapplicable to the placement of decorated capitals at the start of lines in the poem. If we eliminate all solutions which contain non-integral, zero, or negative values, and insist that the sequence increases monotonically from left to right.

$$a < b < c < d < e < f < g < h < i < j < k < l < m < n$$

where a=1, m=1357, and l=1813, we ensure a finite set of solutions. However, for example, setting C_1=-251, C_2=-488, C_3=-604, C_4=-894, and C_5=-1160 yields the sequence b=126, c=198, d=251, e=346, f=488, g=557, h=604, i=691, j=779, k=894, and l=1160, which fulfills all the eleven requirements listed by Crawford, but differs from the location of the capitals in *Cleanness*. This is sufficient to demonstrate that this restricted set contains more than one acceptable solution.

Our aims now are firstly, to determine just how many acceptable solutions there are in this restricted set, and, more importantly, just how likely is it to hit upon an acceptable solution purely by chance. Of course setting C_1=-601, C_2=-249, C_3=-485, C_4=-893, C_5=-1157, negatives of five capital locations in *Cleanness*, necessarily produces a single solution which is the whole range of decorated capitals in the poem. We now define a (5 dimensional) space to search in terms of a single parameter, "*delta (δ)*" which restricts the above coefficients, C_n etc, to be integral values varying between $C_n \pm \delta$. Thus for δ=1 the position of these five capitals could be moved one line backwards or forwards, 3 possibilities for each in all. Applied to all five coefficients there are 243 different possible combinations, a solution space given by 3 raised to the power 5. The size of the solution space grows rapidly with increasing delta, for δ=20 there is a space of 115,856,201 possible solutions. The smallest gap between decorated capitals in the poem is the 44 lines between the capitals at line 556 (g) and 600 (h), so we set an upper limit of 20 on delta to restrict solutions to the ordering requirement above.[iv] For delta>13 we do get a little re-ordering of two pairs of lines (29 in fact for δ=14), but we reject these solutions. The total numbers of completely acceptable results are shown in the Table 5, "Solution Space for δ=±0-20" and Figure 1, "Counts of Integral Solutions for delta from 1 to 20".

[iv]While this places a restriction on the positions of the five capitals determined by the constants, the other capital positions are freer to wander. For example, for δ=5, in the extreme case, the capital at line position 129 only varies by ±2 (123-127), but the capital at 193 varies between ±15 (178-208). See Figure 2, "Distribution of Counts of Integral Solutions for δ=5".

Table 5. Solution Space for δ=±0-20

delta[a]	Number of Solutions[b]	size of space[c]	success rate[d]
0	1	1	100.000
1	27	243	11.111
2	525	3,125	16.800
3	1,911	16,807	11.370
4	8,505	59,049	14.403
5	18,755	161,051	11.645
6	50,869	371,293	13.701
7	89,775	759,375	11.822
8	189,873	1,419,857	13.373
9	295,659	2,476,099	11.941
10	538,461	4,084,101	13.184
11	773,927	6,436,343	12.024
12	1,275,625	9,765,625	13.062
13	1,734,291	14,348,907	12.087
14	2,661,736	20,511,149	12.977
15	3,466,537	28,629,151	12.108
20	14,029,763	115,856,201	12.110

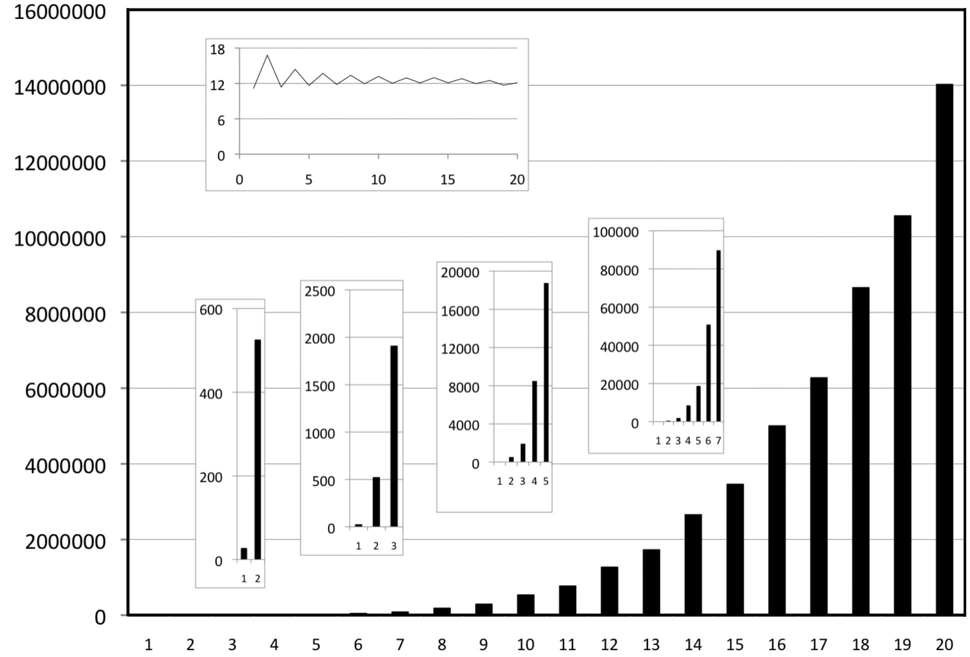
[a]delta is the maximum variation about each of the positions of the decorated capitals in *Cleanness*. E.g. for δ=3, the position of the capital at line 557 can vary between lines 554 and 560. When δ=0 this forces an exact agreement of the single solution with the values found in *Cleanness*.

[b]The number of solutions conforming to the 11 primary criteria listed by Crawford.[CRAWFORD93] These are restricted to solutions with integral line numbers, and the many solutions with non-integral line numbers, negative or zero values, or increasing from b to l are all excluded.

[c]The size of the space searched for solutions is (2*delta+1)**5. E.g. for delta=3, (2*δ+1)=7 and the space searched is 7**5=16807.

[d]The percentage of the space searched that results in a fit to the 11 criteria including the additional constraints. The odds against picking any particular solution by chance is given bythe reciprocal of the average of these percentages multiplied by 100. E.g. For an average of 12%, the odds are 8 to 1 against placing the capitals at these positions purely by chance.

Figure 1. Counts of Integral Solutions for delta from 1 to 20

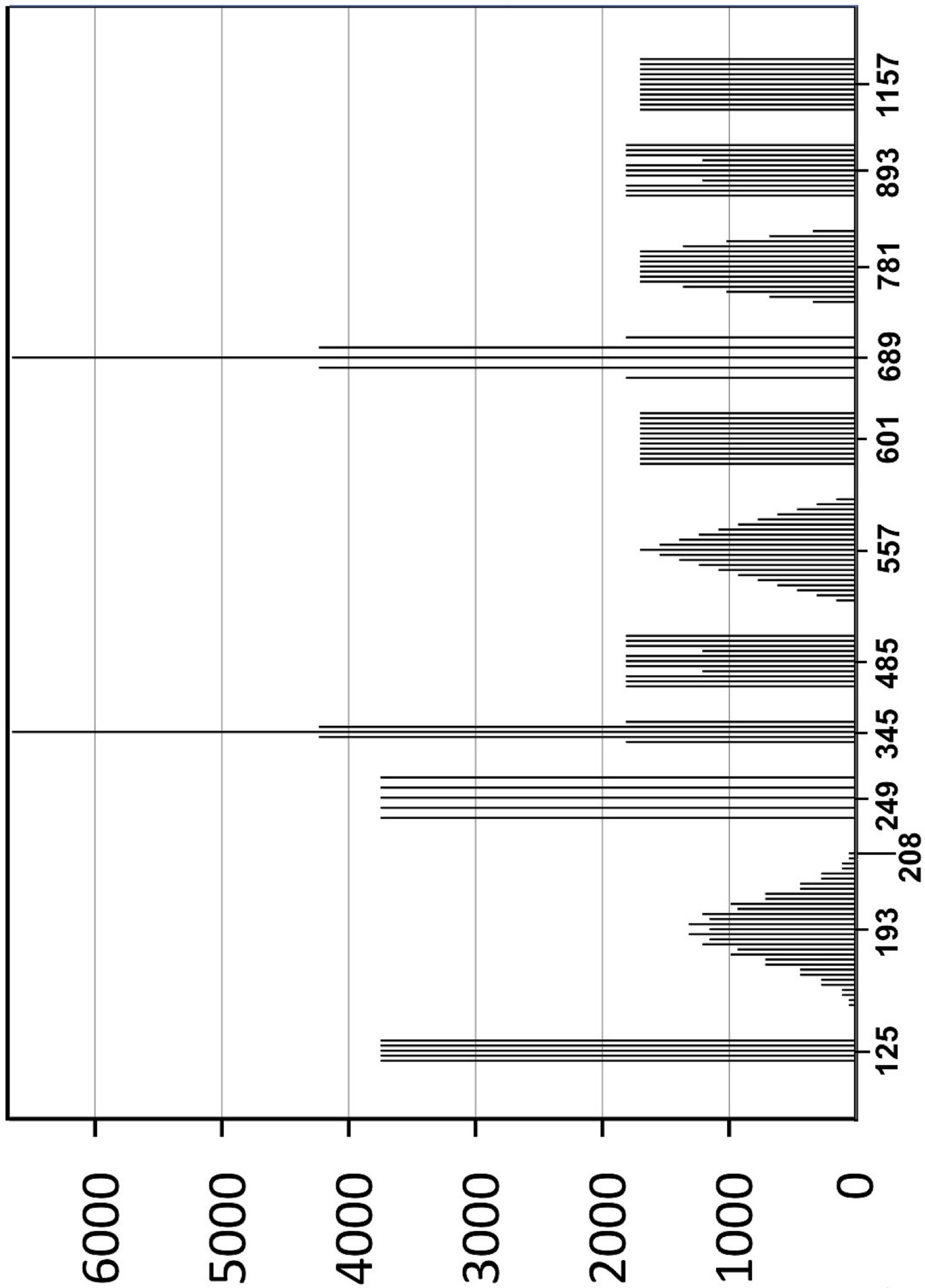


The abscissa is the value of delta. The ordinate is the size of the solution space.

With over fourteen million acceptable solutions resulting from moving decorated capitals up to twenty lines backwards or forwards it is clear that the positions seen in *Cleanness* are by no means unique. More importantly, the success rate converges to about 12% or only about 8 to 1 against meeting all the requirements by chance alone - see insert on the figure. The possibility of choosing a solution which meets all the requirements listed by Crawford by chance alone is very little less than rolling a six with a die: a 16.7% chance of a random success, or 6 to 1 against.

One might perhaps suspect that despite the high number of possibilities, there could be some tendency to favour locations for a decorated capital. To investigate this a little, we determined the counts for each line position that occurred for a range of ±5. The results in Figure 2, “Distribution of Counts of Integral Solutions for δ=5” show that capitals at 345 and 689 are strongly favoured, and there is clearly strong correlation between pairs of lines at 485 and 893, 601 and 1157, and 125 and 249. It is perhaps interesting that there are no possibilities for decorated capitals at even-numbered lines near 249 and 689.

Figure 2. Distribution of Counts of Integral Solutions for δ=5



If we allow a little text-related choice by the scribe, say the capital at line 601 which both Spearing, and Andrew and Waldron, agree corresponds to the start of an important textual passage (601-1048), then the odds against locating the other decorated capitals purely by chance is, rather surprisingly, unchanged. Fixing h=601 does reduce the number of possible solutions, but the percentage of integral solutions is exactly the same. If instead we agree that the scribe could have also located the capital at line 249 (d) introducing the passage 249-544 recognised by both authorities, then the odds are reduced to exactly 3 to 1 for a space of δ=1. Although fixing both d=249 and h=601 produces fewer solutions (9,261), the percentage of integral solutions is the same as for fixing d=249 alone. the results are presented in Table 6, “Percentages of integral, non-negative and non-zero solutions”.

Table 6. Percentages of integral, non-negative and non-zero solutions

delta	Unrestricted		fix d=249		fix h=601		fix d=249 and h=601	
	Number of solutions	% integral	Number of solutions	% integral	Number of solutions	% integral	Number of solutions	% integral
1	243	11.11	81	33.33	81	11.11	27	33.33
2	3,125	16.80	625	28.00	625	16.80	125	28.00
3	16,807	11.37	2,401	26.53	2,401	11.37	343	26.53
4	59,049	14.40	6,561	25.93	6,561	14.40	729	25.93
5	161,051	11.65	14,641	25.62	14,641	11.65	729	25.62
10	4,084,101	13.18	194,481	25.17	194,481	13.18	9,261	25.17

So far we have allowed all capital line positions to vary by the same amount, δ=1,2 … 20. For completion we looked at varying each of the five constants independently and in groups of two, three, and four. Fixing h=601, f=485, k=893, and l=1157, but allowing d to vary as 249±2 produces five solutions, all of which are integral, 100% success. Fixing h, d, f, and k but allowing l=1157±2 also results in 100% success. Fixing d=249, h=601, and l=1157 and allowing (separately) either f=485±2 or k=893±2 produces in both cases a single integral result identical with the values in *Cleanness*. Allowing both f and k to vary together, but fixing the others, is also effective in increasing the number of valid integral solutions, suggesting a strong correlation between the values of f and k. See Table 7, “Integral Solutions for delta=5 applied to Individual Constants”.

Table 7. Integral Solutions for delta=5 applied to Individual Constants

h=601 delta1	d=249 delta2	f=485 delta3	k=893 delta4	l=1157 delta5	total solu-tions	integral solu-tions
2	0	0	0	0	5	3
0	2	0	0	0	5	5
0	0	2	0	0	5	1
0	0	0	2	0	5	1
0	0	0	0	2	5	5
2	2	0	0	0	25	15
2	0	2	0	0	25	5
2	0	0	2	0	25	5
2	0	0	0	2	25	25
0	2	2	0	0	25	3
0	2	0	2	0	25	3
0	2	0	0	2	25	15
0	0	2	2	0	25	7
0	0	2	0	2	25	***
0	0	0	2	2	25	5
2	2	2	0	0	125	15
2	2	0	2	0	125	15
2	2	0	0	2	125	15
2	0	2	2	0	125	35
2	0	2	0	2	125	25
2	0	0	2	2	125	25
0	2	2	2	0	125	21
0	2	2	0	2	125	15
0	2	0	2	2	125	15
0	0	2	2	2	125	35
2	2	2	2	0	625	105
2	2	2	0	2	625	75
2	2	0	2	2	625	75
2	0	2	2	2	625	175

0	2	2	2	2	625	105
2	2	2	2	2	3125	525

This pairwise correlation of the total counts for each variation of line position is also shown in Figure 2, “Distribution of Counts of Integral Solutions for δ=5” for all the integral solutions for δ=5. The correlations between pairs of line positions are clearly evident from the shapes of the distribution patterns: 125 and 249, 485 and 892, 601 and 1157, and, 345 and 689.

It is becoming very hard to deny that the complete sequence of decorated capitals could have been achieved purely by chance. Indeed the comment by Spearing that “*I think that the scribe used decorated initials with intelligence, but somewhat freely and without making a very close study of the parts into which the poem falls*” seems particularly apt.

5. Integral Ratios of Differences

5.1. Integral Ratios of Differences in *Cleanness*

The analysis by Crawford was based upon the differences between the line positions of the capitals in *Cleanness* (relations (1) to (9). For example (h-c)=408 and(k-f)=408. The analysis was further extended to include two ratios of differences which were described as multiples: (k-f)=(h-c)=408 is twice (k-i)=(i-f)=204 and (l-b)=(n-j)=1032 is three times (e-a)=(i-e)=344. These we expressed symbolically in relations (10) and (11), where they were shown to contribute no further information. Relations (10) and (11) above are but two particular examples of the general case where ratios of differences are small integers (2 and 3 respectively). There are 4,095 unique ratios between the differences in *Cleanness* Table 1, “Difference Table”, but only 83 of these are integer ratios. These are listed in Table 8, “Integral ratios of differences for *Cleanness*”. Of these, 29 are very high multiples (greater than 5) and might possibly be ignored, whilst the 9 that have a ratio of 1 have already been covered in relations (1) to (9). The remaining 45 are ratios between 2 and 5 inclusively, and could possibly contribute more information. Although the two ratios noted by Crawford can be derived from the six primary relations, many of these integral ratios cannot. For example, adding the set of four below (chosen almost at random from the 45 to include 10 of the 11 variables) to the basic six ratios equal to 1

(k-j)=2*(d-c)	⇒	2*c-2*d-j+k=0	..........(18)
(l-f)=3*(j-g)	⇒	f-3*g+3*j-l=0	..........(19)
(i-c)=4*(d-b)	⇒	4*b-c-4*d+i=0	..........(20)
(k-c)=5*(f-e)	⇒	c-5*e+5*f-k=0	..........(21)

gives rise to a perfectly acceptable solution, identical with the values observed in *Cleanness*. Reducing the number of added equations to two gives the expected increase in the number of solutions. Furthermore, replacing all of the relations (1), (2), (3), (4), (6), and (8) with high integral ratio relations produces exactly the same results as the basic six. This is hardly surprising as the difference table Table 1, “Difference Table” is built out of the positions of the capitals in *Cleanness* and includes all the original eleven relations noted by Crawford as a subset.

Table 8. Integral ratios of differences for *Cleanness*

(d-b)=124/(b-a)=124 ratio=1	(i-f)=204/(c-b)=68 ratio=3	(k-f)=408/(c-b)=68 ratio=6
(i-e)=344/(e-a)=344 ratio=1	(i-g)=132/(h-g)=44 ratio=3	(k-g)=336/(d-c)=56 ratio=6
(k-f)=408/(h-c)=408 ratio=1	(k-b)=768/(h-e)=256 ratio=3	(l-f)=672/(k-j)=112 ratio=6
(k-h)=292/(f-c)=292 ratio=1	(k-g)=336/(k-j)=112 ratio=3	(l-k)=264/(h-g)=44 ratio=6
(k-i)=204/(i-f)=204 ratio=1	(k-i)=204/(c-b)=68 ratio=3	(m-j)=576/(e-d)=96 ratio=6
(l-g)=600/(h-a)=600 ratio=1	(l-b)=1032/(e-a)=344 ratio=3	(g-d)=308/(h-g)=44 ratio=7
(l-h)=556/(g-a)=556 ratio=1	(l-b)=1032/(i-e)=344 ratio=3	(h-b)=476/(c-b)=68 ratio=7
(n-j)=1032/(l-b)=1032 ratio=1	(l-f)=672/(j-g)=224 ratio=3	(k-d)=644/(j-i)=92 ratio=7
(n-l)=656/(j-b)=656 ratio=1	(l-g)=600/(m-l)=200 ratio=3	(l-e)=812/(h-f)=116 ratio=7
(c-a)=192/(e-d)=96 ratio=2	(l-k)=264/(i-h)=88 ratio=3	(l-f)=672/(e-d)=96 ratio=7
(d-a)=248/(b-a)=124 ratio=2	(m-j)=576/(c-a)=192 ratio=3	(h-d)=352/(h-g)=44 ratio=8
(d-a)=248/(d-b)=124 ratio=2	(n-j)=1032/(e-a)=344 ratio=3	(k-b)=768/(e-d)=96 ratio=8
(f-b)=360/(j-h)=180 ratio=2	(n-j)=1032/(i-e)=344 ratio=3	(m-j)=576/(g-f)=72 ratio=8
(h-c)=408/(i-f)=204 ratio=2	(n-m)=456/(e-c)=152 ratio=3	(n-c)=1620/(j-h)=180 ratio=9
(h-c)=408/(k-i)=204 ratio=2	(h-d)=352/(i-h)=88 ratio=4	(i-d)=440/(h-g)=44 ratio=10
(i-a)=688/(e-a)=344 ratio=2	(i-c)=496/(b-a)=124 ratio=4	(n-k)=920/(j-i)=92 ratio=10
(i-a)=688/(i-e)=344 ratio=2	(i-c)=496/(d-b)=124 ratio=4	(f-a)=484/(h-g)=44 ratio=11
(i-c)=496/(d-a)=248 ratio=2	(j-g)=224/(d-c)=56 ratio=4	(m-b)=1232/(k-j)=112 ratio=11
(i-d)=440/(e-b)=220 ratio=2	(k-b)=768/(c-a)=192 ratio=4	(m-e)=1012/(j-i)=92 ratio=11
(i-h)=88 /(h-g)=44 ratio=2	(m-b)=1232/(g-d)=308 ratio=4	(l-f)=672/(d-c)=56 ratio=12
(j-g)=224/(k-j)=112 ratio=2	(m-g)=800/(m-l)=200 ratio=4	(m-b)=1232/(i-h)=88 ratio=14
(k-f)=408/(i-f)=204 ratio=2	(m-k)=464/(h-f)=116 ratio=4	(l-a)=1156/(c-b)=68 ratio=17
(k-f)=408/(k-i)=204 ratio=2	(e-b)=220/(h-g)=44 ratio=5	(n-d)=1564/(j-i)=92 ratio=17
(k-j)=112/(d-c)=56 ratio=2	(f-b)=360/(g-f)=72 ratio=5	(m-b)=1232/(d-c)=56 ratio=22
(l-f)=672/(k-g)=336 ratio=2	(i-d)=440/(i-h)=88 ratio=5	(m-e)=1012/(h-g)=44 ratio=23
(l-k)=264/(i-g)=132 ratio=2	(k-c)=700/(f-e)=140 ratio=5	(n-d)=1564/(c-b)=68 ratio=23
(m-f)=872/(j-e)=436 ratio=2	(g-b)=432/(g-f)=72 ratio=6	(m-b)=1232/(h-g)=44 ratio=28
(h-a)=600/(m-l)=200 ratio=3	(h-c)=408/(c-b)=68 ratio=6	

If we select six of the ratios greater than or equal to ten, involving some of the set of variables b, c, … l and possibly the constants a=1 and n=1813, but never m=1357, we have another set of six equations in eleven variables

(n-d)=23*(c-b)	⇒	23*b-23*c-d=-n	..........(22)
(n-d)=17*(j-i)	⇒	23*b-23*c-d=-n	..........(23)
(l-a)=17*(c-b)	⇒	-17*b+17*c-l=-a	..........(24)
(l-f)=12*(d-c)	⇒	12*c-12*d-f+l=0	..........(25)
(f-a)=11*(h-g)	⇒	-f-11*g+11*h=-a	..........(26)
(n-k)=10*(j-i)	⇒	10*i-10*j-k=-n	..........(27)

which gives a result identical to that of the initial set of six relations, (1), (2), (3), (4), (6), and (8), but I seriously doubt if anyone would claim that the Gawain-Poet designed a frame for *Cleanness* involving high ratios such as (n-d)=23*(c-b) of (22). Out of the set of 83 integral ratios, any set of 12 (96,851,050,940,000 of them!) produces the exact distribution of capitals in *Cleanness*. Choosing a set of any 6 (377,447,148 of them!) produces exactly the same results as choosing relations (1), (2), (3), (4), (6), and (8).

The question that now arises is “how much weight can we put on higher multiples of differences, i.e. ratios greater than 1?”. We have more than ample ratio relations to insist that the only, unique, solution is that in *Cleanness*. In effect, if we were to allow the use of higher multiples, we would be asking “how many times can the exact sequence 1, 125, 193, 249, 345, 485, 557, 601, 689, 781, 893, 1157, 1357, 1813 occur”. Not surprisingly the answer is “only once”. We are faced with a choice, either the sequence in *Cleanness* is absolutely unique (as Crawford claimed) or we cannot use integral ratios greater than 1 (equalities) without over-determining the solution. To accept all 83 integral ratios as definitive implies that they must have been explicitly designed in by the Gawain-Poet. This extent of design work is hardly feasible, and we prefer to reject all ratios greater than 1.

5.2. Integral Ratios of Differences in an Expanded Space

If we extend the treatment of integral ratios of differences to the fourteen million other solutions we found in Table 5, “Solution Space for δ=±0-20”, we see that there are always sufficient higher ratios to insist that each of the solutions is unique and conforms equally well to the eleven Crawford criteria. The inference must be that unless we insist that relations such as (n-d)=23*(c-b) of (22) are vital to the structure of *Cleanness*, then we are not justified in using ratios greater than one. Intuitively this appears reasonable. However, looking at the other poems in the Nero A.x manuscript for a moment, there are 21 decorated capitals in *Pearl* equally spaced at intervals of 60 lines 1, 61, 121, … 841 with a slight hiccup at section XV, 913 to 961 then 1033, 1093, 1153. In addition to these decorated capitals there occurs a small, slanting double line in the left hand margin at the start of every group of twelve lines, dividing each 60-line section into 5 stanzas of 12 lines each. Although the Gawain-Poet might have insisted on a discipline of repetition of groups of 12 lines (stanzas) within the 20 sections of 60 lines each (except the anomalous XV which contains 6 12-line stanzas), there is no sign of decorated capitals every 12 lines (to make a ratio of 60/12=5.) He never attempted to introduce sections de-marked by decorated capitals of exactly double or triple length (and why would he want to do such a thing anyway?). Similarly, in *Cleanness*, there is the same evidence in the Nero A.x manuscript that *Cleanness* has a quatrain structure with marginal marks every four lines, and some editors have presented the poem with this quatrain structure emphasised.([ANDERSON77] The left margin marks appear again in *Patience* every four lines and in *Sir Gawain and the Green Knight* at the first long line following the rhyming wheel, introducing sections varying in length form 12 lines (20-31) to 25 lines (536-560). It is also feasible that these marginal marks were simply a tallying method used by the scribe and not a structure integral to the poems. Even if we were to accept these marginal marks as defining higher ratios, they only very rarely coincide with the decorated capitals in any of the poems

My inclination is to ignore any integral ratios greater than 1. The ratio 1 relations clearly de-mark sections of the poem (whether intentionally or not), but multiple relationships do not have any obvious purpose. Why should the Gawain-Poet have wanted the section (k-f)=408 to be exactly twice as long as (k-i)=204, what purpose could it serve, and what meaning might it have in a textual or symbolic context? We remain convinced that only of a set of five constants is necessary to completely define the eleven criteria listed by Crawford.

6. The Golden Ratio

We now consider the two relations noted by Crawford, the approximate equality of the ratios of differences between line positions to the Golden Ratio (g-a)/(e-a)≅(√5+1)/2=1.618033989 and the function (b-a)/(h-a)≅(√2-1)/2=0.207106781. We will refer to the latter as the "root 2 function" where R2f. So far we have been dealing with integers where any decisions are clear-cut, either (k-h) is equal to (f-c) or it is not, and for all the fourteen million solutions we found (k-h) does equal (f-c) exactly. However we noted in the previous section that only 83 of the 4,095 ratios are integral, and the remaining 4,012 ratios are non-integral. Crawford proposed that two of these non-integral but rational ratios were of major significance to the structure of the poem, and indeed were the ones that defined the geometrical structure attributed to *Cleanness*. The Golden Ratio was apparently first defined by Euclid in Book VI of the "*Elements*" and was used in the construction of the regular pentagon.[v] The value of the irrational Golden Ratio is the positive root, x=(1+√5)/2 of the quadratic equation x^2-x-1=0 and takes the approximate value of 1.6180339887…[vi] Crawford chose the negative root, x=(1-√5)/2, of the quadratic, ignored the negative sign, and gave the value of the Golden Ratio as (√5-1)/2=0.618[vii]. Noting that the ratio of (e-a)=344 and (g-a)=556 (0.618705036) is close to the reciprocal of the Golden Ratio (0.618033989), Crawford proposed that the Golden Ratio is an important feature of the structure of *Cleanness*. Similarly she noted that the ratio of (b-a)=124 and (h-a)=600 (0.206666667)is close to (√2-1)/2=0.207106781, and that the square root of two is the length of the diagonal of a unit square. With these two ratios established, together with the sum of 344 and 556 (900) she was able to derive a geometrical construction of the positions of all the decorated capitals in *Cleanness*. The construction was elegant and in the true Euclidean tradition (and well worthy of praise and a medal from Euclid), but just as Euclid knew where he was going in the construction of the pentagon, so Ms Crawford knew she was aiming at the positions of the capitals in *Cleanness*. The Gawain-Poet on the other hand, if he did design a frame for this poem, was not working towards a known objective, and there is very little chance that he could have hit upon this construction while driving blind.

The ratio 556/344 (1.61627907) is reminiscent of the Golden Ratio, but the critical point here is "*is it close enough to be considered significant*" or "*how good is good enough*"? This has to be a matter of personal judgement, there is no objective criterion to insist that 1.61627907 and 1.618033989 are close enough (the difference is 0.001754919) to be considered equal.[viii] We need to enquire how the frequency of occurrence of approximations to the Golden Ratio varies with accuracy, and what level of precision we should require if we are to claim identity. We might also

[v]Euclid made no claim to have discovered the Golden Ratio, but his was the first formal definition as far as we know. The Golden Ratio also defines the pentagram which is so important in *Sir Gawain and the Green Knight*, also by the author of *Cleanness*, the Gawain-Poet.

[vi]The value of the Golden Ratio has been calculated to at least 10 million decimal places, and Mario Livio [LIVIO02] p. 81 quotes it to 2,000 decimal places.

[vii]The Golden Ratio is (√5+1)/2=1.618033989, it's reciprocal is 2/(√5+1)=0.618033989. The negative root of the quadratic equation x^2-x-1=0 is x=(1-√5)/2=-0.618033989.

[viii]Crawford quoted the ratio 344/556 (=0.618705036) which looks a lot closer to the reciprocal of the Golden Ratio, 0.618033989, (the difference is 0.000671047), but the relative accuracy in both cases is the same, 0.001085.

note that both √5 and √2 are irrational numbers, and they also *cannot* be represented by the ratio of any integers (they also caused much heart-searching among the Pythagoreans).

There are a few ways of introducing a little objectivity into the decision about what is "*good enough*".

• Do any other basic quantities (π or √11 for example) occur with similar (or better) accuracy?
• Does the Golden Ratio occur elsewhere in *Cleanness*, possibly to higher accuracy than 556/344?
• Do the Golden Ratio and the root 2 function occur in the other solutions?
• What is the textual or thematic significance of line 900 in *Cleanness* (the sum of the differences 556 and 344 whose ratio is supposedly close enough to the Golden Ratio)?

If we can answer "yes" to the first three questions and find little significance in line 900, then our confidence in the importance of the Golden Ratio and the √2 function must be seriously diminished. We now proceed to explore these questions.

In a search for other "basic quantities" there is obviously much scope for bias and there is always the danger of choosing one that fits best. Of the two noted by Crawford, one, the Golden Ratio, is a fundamental constant and the other, the root 2 function, includes the length of the diagonal of the unit square. Taking a similar path, we look for the even better known constant, π, and the length of the diagonal of the unit cube, √3. The square root of 5 determines the Golden Ratio, so we include √5 in our search. To round out our search we include all the other irrational square roots up to 15, noting that the square rots of 4, 9, and 16 have already been found as integral multiples or ratios in Table 8, "Integral ratios of differences for *Cleanness*". Perhaps of primary importance is the accuracy with which a ratio of integers approximates to the value of an irrational constant, and we explore this first for the ratios of line positions of the capitals in *Cleanness* which approximate to the Golden Ratio and the root 2 function and various other constants in Table 9, "Dependence of the occurrences of the Golden Ratio (GR), the root2 function (R2f), and other constants in *Cleanness* on the relative accuracy (relacc) required for a successful test" below.

Table 9. Dependence of the occurrences of the Golden Ratio (GR), the root2 function (R2f), and other constants in *Cleanness* on the relative accuracy (relacc) required for a successful test

relacc	GR	R2f	√2	√3	√5	√6	√7	√8	√10	√11	√12	√13	√14	√15	π
0.00005	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.00006	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0
0.0001	0	0	0	1	0	0	0	0	0	0	1	1	2	0	0
0.0003	0	0	2	1	0	0	3	0	0	0	1	3	2	0	0
0.0005	0	0	3	2	3	1	5	0	1	2	3	5	3	2	2
0.0007	0	0	3	2	3	1	5	2	1	2	3	5	6	2	2
0.001	1	0	5	3	7	1	6	5	1	3	4	5	7	2	3

0.002	8	4	12	11	11	7	7	12	3	3	5	5	7	3	8
0.003	14	9	14	15	17	10	14	16	5	6	8	9	10	4	12
0.004	19	11	20	18	18	14	19	22	14	14	10	12	15	9	18
0.005	24	11	23	20	26	18	22	23	14	14	10	12	15	9	18
0.007	34	14	36	34	33	20	29	31	21	22	23	19	21	17	23
0.010	59	21	49	47	51	30	40	43	27	36	31	24	30	31	30

At a relative accuracy of 0.0005 we find reasonable approximations to √2, √3, √5, √6, √7, √8, √10, √11, √12, √13, √14, √15, and π, but no approximations to either the Golden Ratio or the root 2 function so important to the geometrical interpretation of Crawford. We now extend the treatment to the additional solutions we found in the expanded spaces. In Table 10, “Acceptable Non-Intgral Ratios as a function of accuracy (relacc) and space (delta)” we explore the occurrence of approximations to the Golden Ratio, the root 2 function, √2, √3, √5, √11, and π for spaces with line positions expanded by ±1 to ±5 lines

Table 10. Acceptable Non-Intgral Ratios[a] as a function of accuracy (relacc) and space (delta)

delta[b]	±1 (243 solutions)				±2 (3,125 solutions)				±3 (16,807 solutions)				±4 (59,049 solutions)				±5 (161,051 solutions)			
relacc[c]	1E-6	1E-5	1E-4	1E-3	1E-6	1E-5	1E-4	1E-3	1E-6	1E-5	1E-4	1E-3	1E-6	1E-5	1E-4	1E-3	1E-6	1E-5	1E-4	1E-3
GR[d]	0	0	6	126	0	53	399	2469	0	88	1455	8795	0	214	5460	44217	0	352	11967	101157
R2f[e]	0	0	6	126	0	5	399	2469	0	28	270	3909	0	68	974	16069	0	269	2419	38001
√2[f]	0	6	7	103	0	108	271	1944	0	197	941	8341	0	744	3649	41357	0	1544	7184	98459
√3[g]	0	9	14	123	0	88	313	2501	0	234	1113	9098	0	693	3809	42722	0	1325	8553	93114
√5[h]	0	0	7	188	0	1	302	3336	0	6	1200	11255	0	212	4486	46941	0	389	9571	99612
√11[i]	0	0	0	68	7	7	103	1762	26	26	409	6112	42	42	2952	27965	62	62	6273	60403
PI[j]	0	0	1	99	7	7	287	1794	31	31	873	5857	181	181	2950	25322	336	336	5981	53822

[a]ratios are always taken as greater/smaller, so ratio≥1

[b]delta determines the size of the space to be searched for solutions, e.g for delta=±2 the position of a decorated capital can be between 2 before and 2 after the position in *Cleanness*.

[c]relacc is a relative accuracy, independent of the value to be tested. For example the value to be applied in the test for Golden Ratio is accuracy=relacc*(√5+1)/2 = relacc*1.618033989. The notation 1E-6 denotes a relative accuracy of 0.000001

[d]GR - search for ratios close to the value of the Golden ratio, (√5+1)/2 = 1.618033989

[e]R2f - search for ratios close to the value of the reciprocal of the root 2 function, (√2-1)/2 = 0.207106781, or 4.828427125

[f]√2 - search for ratios close to the value of √2 = 1.414213562

[g]√3 - search for ratios close to the value of √3 = 1.732050808

[h]√5 - search for ratios close to the value of √5 = 2.236067978

[i]√11 - search for ratios close to the value of √11 = 3.316624790. The integer ratio 1257/379=3.316622691 is a very close approximation to √11

[j]PI - search for ratios close to the value of π = 3.141592654. The ingeger ratio 355/113=3.14159292 is a very close approximation to π

At a relative accuracy of 0.0001 (1E-4) we now find approximations to almost everything, including the Golden Ratio and the root 2 function. As the space is expanded we find increasingly better approximations to all the constants, but from ±2 onwards the approximations to π and √11 become significantly better than any of the other constants. For example in the ±5 space we find 336 different solutions all give approximations to π at a relative accuracy of 0.000001 (1E-6): all with (e-a)/(b-a)=355/113=3.14159292[ix] instead of π=3.141592654, a difference of only 0.000000267. For comparison, Crawford decided that (g-a)/(e-a)=556/344=1.61627907 was close enough to the Golden Ratio, 1.618033989, a difference of 0.0017549 or a relative accuracy of 0.0010844, to be accepted as agreement.

Crawford relied upon a relative accuracy of 0.001085 to claim that the Golden Ratio was intentionally built into the capitals of *Cleanness*, but where one draws the line that separates occurrence from non-occurrence without any other evidence is purely subjective. Equating the ratio 556/344 with the Golden Ratio forms the basis of the geometric structure proposed by Crawford: 900, the sum of 556 (g-a) and 344 (e-a), is the critical dimension she proposed for the frame within which *Cleanness* was composed. As Table 11, “Highest accuracy of ratios of differences between the line positions of the decorated capitals in *Cleanness* which are equal to the values of various irrational constants.” shows, the ratio 1468 (n-e) to 908 (l-d) is a better approximation to the Golden Ratio, but unfortunately for the geometric hypothesis the sum, 2376, lies outside the range of the poem. Using the 556/344 ratio leads to the critical line 900 as the basis of Crawford’s frame for the poem. Line 900 is accorded no significance in the structures identified by Spearing, or Andrew and Waldron, and lies within the instruction to Lot to flee from the destruction of Gomorrah.

‘Wyth þy wyf & þy wyȝez & þy wlonc deȝtters,
For we laþe þe, sir Loth, þat þou þy lyf haue.
Cayre tid of þis kythe er combred þou worþe,
With alle þi here vpon haste, tyl þou a hil fynde; … ’

—The Gawain-Poet, *Cleanness (899-902)*

Surely if the Gawain-Poet placed any significance on (or was even aware of) the importance of line 900 to the structure of the poem, he would have insisted on a clear textual break and a decorated capital at that point.

In Table 11, “Highest accuracy of ratios of differences between the line positions of the decorated capitals in *Cleanness* which are equal to the values of various irrational constants.” we tested the positions of the capitals in *Cleanness* against a variety of other irrational constants, π and square roots, and note the highest accuracy at which we can claim identity.

[ix]Clearly 113 and 355 are outside the ranges of 125±5 and 345±5 respectively, but the general solution derived in Section 4, “*Results*” set C_1=-601, C_2=-249, C_3=-485, C_4=-893, C_5=-1157. In the space search, these constants were restricted to vary by no more than ±5, but this restriction did not apply to other line positions which can, and do, vary by considerably more than ±5.

Only the highest relative accuracy for agreement is given in the table. GR is the Golden Ratio, (√5-1)/2=1.618033989, R2f is the reciprocal (4.828427125) of the root2 function ((√2-1)/2=0.207106781) and relacc is the lowest relative accuracy found which leads to a successful test. The accuracy used in the test is the product of relacc and the true value of the constant to allow valid comparisons between tests of different constants. The tests are listed in order of decreasing accuracy: the best agreement at the top of the table (√3, relacc= 0.00006, difference=0.00009205) is about 19 times better than the agreement with the Golden Ratio used by Crawford (relacc=0.0011, difference=0.00175492).

The most obvious feature is that for an accuracy greater than 0.0011 one can claim agreement for every constant we tested for, and that in every case the agreement is better than that for the Golden Ratio and the root 2 function. With this evidence I believe we must have serious doubts that the Golden Ratio and the root 2 function were built in deliberately to *Cleanness*. We might also note that the square root of 784 (28, also an integer) occurs exactly in *Cleanness*, see Table 8, “Integral ratios of differences for *Cleanness*”. Despite this exact agreement, I question whether the Gawain-Poet deliberately planned that (m-b)=1232 should be exactly 28 times (h-g)=44. It appears that there is a high probability of finding almost any value, *except the Golden Ratio and the root 2 function*, integer, rational or irrational if one is prepared to set the relative accuracy required to less than 0.001085.

Table 11. Highest accuracy of ratios of differences between the line positions of the decorated capitals in *Cleanness* which are equal to the values of various irrational constants.

test for[a]	relacc for 1st. find[b]	details[c]
√3	0.000053	(m-c)=1164/(l-f)=672 ratio=1.73214286 √3=1.73205081 difference=0.00009205
√12	0.000053	(m-c)=1164/(k-g)=336 ratio=3.46428571 √12=3.46410162 difference=0.00018410
√14	0.000074	(m-k)=464/(b-a)=124 ratio=3.74193548 √14=3.74165739 difference=0.00027810
√13	0.000080	(k-e)=548/(e-c)=152 ratio=3.60526316 √13=3.60555128 difference=0.00028812
√7	0.000112	(m-c)=1164/(i-d)=440 ratio=2.64545455 √7=2.64575131 difference=0.00029677
√2	0.000297	(j-b)=656/(m-k)=464 ratio=1.41379310 √2=1.41421356 difference=0.00042046
√15	0.000307	(n-a)=1812/(l-i)=468 ratio=3.87179487 √15=3.87298335 difference=0.00118847
√5	0.000346	(e-c)=152/(c-b)=68 ratio=2.23529412 √5=2.23606798 difference=0.00077386
π	0.000402	(h-d)=352/(k-j)=112 ratio=3.14285714 π=3.14159265 difference=0.00126449
π	0.000402	(i-d)=440/(f-e)=140 ratio=3.14285714 π=3.14159265 difference=0.00126449
√6	0.000432	(k-a)=892/(g-c)=364 ratio=2.45054945 √6=2.44948974 difference=0.00105971
√11	0.000469	(k-h)=292/(i-h)=88 ratio=3.31818182 √11=3.31662479 difference=0.00155703
√10	0.000479	(n-e)=1468/(m-k)=464 ratio=3.16379310 √10=3.16227766 difference=0.00151544
√8	0.000622	(h-a)=600/(g-e)=212 ratio=2.83018868 √8=2.82842712 difference=0.00176155
GR[d]	0.000800	(n-e)=1468/(l-d)=908 ratio=1.61674009 GR=1.61803399 difference=0.00129390

R2f	0.001015	(m-k)=464/(e-d)=96 ratio=4.83333333 R2f=4.82842712 difference=0.00490621
GR[e]	0.001086	(g-a)=556/(e-a)=344 ratio=1.61627907 GR=1.61803399 difference=0.00175492

[a]The irrational constant against which the ratio of the integer line positions in *Cleanness* was tested.

[b]The relative accuracy of the test: "is the absolute value of ratio-(test for) less than relacc*(test for)?". This is the smallest value of relacc for which the test succeeds, i.e. the closest agreement between ratio and the constant.

[c]The differences and their values used to obtain the ratio, the value of the ratio, and the difference from the value tested for).

[d]1468/908 is a better approximation (by about 26%) to the Golden Ratio than the 556/344 noted by Crawford. Unforunately this better approximation does not have the sum of 900 so critical to the geometrical structure proposal.

[e]There are four identical occurrences of the ratio 556/344 used by Crawford: (g-a)/(e-a), (g-a)/(i-e), (l-h)/(e-a), and (l-h)/(i-e). Only one is quoted here, but it is completely immaterial which one is used since g-a=l-h=556 (eq. 5) and e-a=i-e=344 (eq. 1).

We now consider solutions where the line positions differ from those in *Cleanness*. In Table 10, "Acceptable Non-Intgral Ratios as a function of accuracy (relacc) and space (delta)" we give the number of successful matches for a variety of constants for a range of relative accuracies from 0.000001 to 0.1 for solutions in which the line positions were allowed to vary by ±1 from the positions in *Cleanness*. Again we find that several constants occur with an order of magnitude better agreement than do the Golden Ratio and the root2 function. However we did find that The Golden Ratio and the root2 function occurred with an order of magnitude better precision than they do in *Cleanness*. Again we have increased our doubts about the importance of the Golden Ratio in the structure of *Cleanness*.

Table 12. Non-Integral Ratios for the Expanded Space delta=±1

relacc=	0.000001	0.00001	0.0001	0.001	0.01	0.1
GR	0	0	6	126	1538	13610
R2f	0	0	1	67	602	6554
√2	0	6	7	103	1309	13941
√3	0	9	14	123	1319	13455
√5	0	0	7	188	1379	11611
√6	0	3	6	68	946	11204
√7	0	0	1	112	1111	10299
√8	0	6	6	95	1022	10274
√10	0	0	2	62	748	8994
√11	0	0	0	68	930	8754
π	0	0	1	99	874	9047

Finally we look at solutions in the expanded space up to ±5 for a variety of constants and accuracies in Table 10, “Acceptable Non-Intgral Ratios as a function of accuracy (relacc) and space (delta)”. In the extreme case of ±5 and a relative accuracy of 0.000001 we find about 6000 matches for both π and √11, but no matches for either the Golden Ratio or the root2 function.[x]

In conclusion, with the sum of all this evidence I think we must conclude firstly that meeting the Crawford criteria by chance is not too improbable (possibly as low as 3 to 1 against a chance happening), and secondly, given that these criteria are met by chance, any ratios approximating to the Golden Ratio or the root 2 function in *Cleanness* are purely matters of chance (and considerably less likely to occur than π), and cannot have been implemented by design. There is no serious evidence for the tight geometric structure of *Cleanness* proposed by Crawford. The narrative structures proposed by Spearing, and by Andrew and Walton, are well substantiated by the text and a far more realistic approach to design by the Gawain-Poet.

[x]Perhaps it is worthwhile to note that the ratio 355/113=3.14159292 is very close to π=3.1415926535, and that 1257/379=3.316622691 is very close to √11=3.31662479. Again it is worth making the point that this very close agreement in these two cases could never be claimed as evidence of intentional planning; the agreement arises solely because it so happens that π and √11 can be approximated quite closely by ratios of relatively small integers.

7. Conclusions

We find that a set of five completely arbitrary constants, C_1, C_2, C_3, C_4 and C_5 will generate an infinite number of solutions of locations for the decorated capitals in *Cleanness* which all reproduce all the equalities and symmetries noted by Crawford. However, only those resulting in integral line numbers which are greater than zero, are no greater than 1813, and are in ascending order have any meaning in *Cleanness*. Putting C_1=-601, C_2=-249, C_3=-485, C_4=-893 and C_5=-1157 generates the observed line positions of the 11 decorated capitals in *Cleanness*: b=125, c=193, d=249, e=345, f=485, g=557, h=601, i=689, j=781, k=893 and l=1157.

The assertion by Crawford, that the locations of the decorated capitals is unique in the ability to produce the equalities and symmetries she noted, is clearly in error: if we restrict our attention to solutions for which the line positions do not vary by no more than ±20 from the observed line positions in *Cleanness*, we find there are over fourteen million sets of positions which reproduce the criteria exactly. In general we find that whatever range of variation we allow up to ±20, there is a 12% chance of hitting an acceptable solution (Table 5). The odds of locating the decorated capitals so as to reproduce Crawford's criteria purely by chance are about 1 in 8. Not too different from rolling a die.

The construction of the sequence of capitals proposed by Crawford is based upon the occurrence of close approximations to the Golden Ratio and the function of the square root of 2. It is important to consider just how close is 'close enough' to be considered significant. At a relative accuracy of 0.0005 we find reasonable approximations to the irrational constants √2, √3, √5, √6, √7, √8, √10, √11, √12, √13, √14, √15, and π, but *no* approximations to either the Golden Ratio or the function of root 2 (Table 9). With these better, but obviously fortuitous, approximations to many other basic constants, we feel little or no significance can be attached to the poorer approximation to the Golden Ratio and the function of the square root of 2.

We conclude that the equalities and symmetries noted by Crawford are probably no more than chance events, and are not significantly related to the poem or the intentions of the Gawain-Poet.

A. Equalities between Sums of Differences

Two observations of equalities in the line positions which were not noted by Crawford stand out in Table 1, “Difference Table”.

(c-a)+(d-a)=(i-d)	⇒	-c-2*d+i=-2*a	28
(e-b)+(h-g)=(l-k)	⇒	-b+e-g+h+k-l=0	29

The possibility arises that relations such as these could provide additional information about the placement of the decorated capitals in , although they are not past of the scheme proposed by Crawford. On investigation we found that there are a total of 246,221 such sums of two differences, of which 844 show the same pattern of equality as relations (28) and (29) above. Whether we should consider any of these 844 relations as new information to the investigation of alternative line placements is a matter of judgement. It is certainly possible to find far more than sufficient relations between the differences between positions of the capitals to reduce the search to “find me all solutions which are exactly the one we see in ”. The question becomes one of “where do we stop?” My feeling is that these are hardly primary data and are susceptible to fortuitous coincidence, but in Appendix B, *The Decorated Capital at Line 1357* we explore this in more detail with particular regard to the capital at line 3157 which did not appear in Crawfords primary analysis.

B. The Decorated Capital at Line 1357

Surprisingly Crawford did not note any equalities or symmetries involving the last decorated capital in *Cleanness* at line 1357, and only incorporated this line position into her geometric scheme in a rather convoluted manner:

1. If (g-a)/(e-a)=556/344=1.61627907 is close enough to the Golden Ratio (1.618033989), then 344 is to 556 as 556 is to 900.
2. Given 900, construct a 900x900 square and divide it into 9 equal squares 300x300.
3. The diagonal of a 344x300 rectangle is 456.4383858 which is near enough to 456.
4. Add 456 to 900 to get 1356 which is the number of lines preceding the decorated capital at line 1357.

It might seem far simpler to relate m=1357 to the other capitals using integral ratios, although these must all involve ratios greater than one, relations we rejected earlier. However, relaxing this rejection, we find seven such relations relating the capital at line 1357 to the others.

(n-m)=3*(e-c)	⇒	3*c-3*e-m=-n	..........(30)
(m-b)=4*(g-d)	⇒	b-4*d+4*g-m=0	..........(31)
(m-f)=2*(j-e)	⇒	2*e-f-2*j+m=0	..........(32)
(m-j)=3*(c-a)	⇒	-3*c-j+m=-3*a	..........(33)
(m-k)=4*(h-f)	⇒	4*f-4*h-k+m=0	..........(34)
(h-a)=3*(m-l)	⇒	-h-3*l+3*m=-a	..........(35)
(l-g)=3*(m-l)	⇒	-g+4*l-3*m=0	..........(36)

If we build an extended coefficient and vector matrix out of relations (1), (2), (3), (4), (6), (8), (28), (29), (30), (31), (32), (33), (34), (35), and (36) we now have a system of 15 equations in 12 variables (b,c,…m) which is over-determined and will only produce a unique solution identical with the positions of the capitals in *Cleanness*.

Thus the matrix A is now

Table B.1. Coefficient matrix - 15 equations in 12 variables

	0	0	0	-2	0	0	0	1	0	0	0	0
	-2	0	1	0	0	0	0	0	0	0	0	0
	0	0	0	0	1	0	0	-2	0	1	0	0
	0	0	0	0	0	-1	-1	0	0	0	1	0
	1	0	0	0	0	0	0	0	-1	0	-1	0
	0	1	0	0	-1	0	-1	0	0	1	0	0
A =	0	-1	-2	0	0	0	0	1	0	0	0	0
	-1	0	0	1	0	-1	1	0	0	1	-1	0
	0	3	0	-3	0	0	0	0	0	0	0	-1
	1	0	-4	0	0	4	0	0	0	0	0	-1
	0	0	0	2	-1	0	0	0	-2	0	0	1
	0	-3	0	0	0	0	0	0	-1	0	0	1
	0	0	0	0	4	0	-4	0	0	-1	0	1
	0	0	0	0	0	0	-1	0	0	0	-3	3
	0	1	0	0	0	0	0	0	0	-1	-3.5	3.5

and the constant vector C is now

Table B.2. Vector of constants

	-1
	-1
	0
	-1
	-1813
	0
C=	-2
	0
	-1813
	0
	0
	-3
	0
	-1
	0

Using this system of linear equations we can now investigate how the number of solutions varies with the number of equations, the number of unknowns, and the space to be searched. This is shown in Table B.3, "Integral solutions as a function of number of equations and number of variables".

Table B.3. Integral solutions as a function of number of equations and number of variables

delta[a]	6_11[b]	7_11	8_11	9_12	10_12	11_12	12_12
1	27/243[c]	9/81	1/27	1/27	1/9	1/3	1/1
2	525/3125	45/625	3/125	1/125	1/25	1/5	1/1
3	1911/16807	133/2401	5/343	3/343	1/49	1/7	1/1
4	8505/59049	477/6561	15/729	3/729	1/81	1/9	1/1
5	18755/161051	869/14641	25/1331	7/1331	1/121	1/11	1/1
7	89775/759375	2985/50625	63/3375	21/3375	1/225	1/15	1/1
10	538461/4084101	12789/194481	197/9261	51/9261	1/441	1/21	1/1
15	3466537/28629151	56017/923521	575/29791	179/29791	7/961	1/31	1/1
20	14029763/115856201	181261/2825761	1419/68921	369/68921	9/1681	1/41	1/1

[a]delta is the maximum allowed variation on the line position in *Cleanness*. ±3 allows the capital to move by plus or minus 3 lines - a space of 7 possibilities

[b]6_11 means 6 equations in 11 variables

[c]27/243 means 27 solutions found in a space of 243.

As the space is increased, the number of solutions increases. As the number of equations is increased, the number of solutions decreases. As the number of variables increases, the number of solutions increases. There always comes a point when we can force a unique and exact agreement with the line positions of the decorated capitals in *Cleanness*, but this is a tautology, this is the answer I want, find it for me. As a well-known statistician once said "if you torture the numbers long enough they will eventually give you the answer you want".

I believe we should exclude relations (30), (31). (32), (33), (34), (35), and (36) on the grounds discussed earlier in Section 5.1, "*Integral Ratios of Differences in Cleanness*". There are just too many higher integral ratios, and some at least can be derived from ratios of unity. I also believe we are not justified in including relations (28) and (29), they could well be simple coincidences. So we are left with the integral relations noted by Crawford, but excluding the non-integral ratios supposedly close to the Golden Ratio and the root 2 function, and with these alone we have found some 14 million equally good solutions that satisfy the eleven (reduced to only six) relations for the locations of the decorated capitals.

C. Computer Codes

Table of Contents

The results reported in this paper were obtained using ten separate computer codes. Copies of these are available on request from ron@catterall.net. The codes are listed below in Table C.1, “Program Codes used in this work” with some limited description.

The codes were written in standard Perl and run on Perl version 5.8.6 (patched) on a Power PC G4 Macintosh with MaC Os X 10.4.11, and on Perl version 5.12.3 running on Cygwin 1.7.9-1 on Windows 7 (64bit). The coding has been kept as simple as possible without reliance on the many short-cut variables provided by Perl. There is no dependence on libraries available from CPAN, and any standard Perl implementation will suffice. Some of the numerical routines (standard matrix algebra) are direct translations from FORTRAN and the style shows their age, but they are well tested and fast. The logic of the programs should be easy to follow, they and should run on almost any platform.

In brief, the programs of the form 'lineq_6_11_5.pl' perform the primary search for solutions. In this case for 6 equations describing 11 variables with a variation (delta) of 5. The primary output is written to a file 'lineqoutput_6_11_5.txt' which is human readable.

The other programs read this output file and produce reports, for example the program 'count' produces a file, 'count_output_6_11_5.txt', containing the counts of the number of times a decorated capital is placed at each line number - the data in Figure 2, “Distribution of Counts of Integral Solutions for δ=5”.

The numbers of the equations (6), variables (11) and delta (5) in the example above can be easily changed at the start of the program. Beware though, the files can be large with hundreds of millions of records.

C.1.

Table C.1. Program Codes used in this work

Prograqm	Description
marank	Determine the rank of a matrix. This code is built into the lineq codes
lineq_6_11	The basic code to solve for the 11 relations noted by Crawford

lineq_6_10d	As 6-11, but fix d=249
lineq_6_10h	As 6_11, but fix h=601
lineq_6_9dh	As 6_11, but fix d=249 and h=601
check	Check output from lineq_6_11 against Crawford's relations and also look for and remove any line order reversals at high delta.
count	Count total occurrences of each line number
rats	Integral ratios in *Cleanness*
mrats	Explore all possible ratios of 2 line positions against many constants
pirats	Detailed exploration of the occurrence of π

References

[ANDERSON77] J J Anderson. *Cleanness*. (1977). Manchester University Press. Barnes & Noble, New York.

[ANDREW02] Malcolm Andrew and Ronald Waldron. *The Poems of the Pearl Manuscript*. Pearl, Cleanness, Patience, Sir Gawain and the Green Knight. (2002). University of Exeter Press.

[CRAWFORD93] Donna Crawford. *The Architectonics of "Cleanness"*. Studies in Philology. 90. 29-45. (1993).

[EDWARDS97] A S G Edwards. *The Manuscript: British Library MS Cotton Nero A.x*. 197-219. *A Companion to the Gawain-Poet*. ed. Derek Brewer and Jonathan Gibson. (1997). D.S.Brewer, Brewer & Brewer Ltd. Woodbridge, Suffolk 1P12 3DF, UK.

[GOLLANCZ21] Sir. Israel J Gollancz. *Purity*. (1921). . .

[HILL46] Laurita Lyttleton Hill. *Madden's Divisions of Sir Gawain and the "Large Initial Capitals" of Cotton Nero A.X.*. Speculum. 21. 67-71. (1946).

[LIVIO02] Mario Livio. *The Golden Ratio*. (2002). Broadway Books. New York. The most recent and most systematic study of the mathematics of Φ and the supposed role of the Golden Ratio in the arts generally..

[MADDEN39] The first published version of Sir Gawain and the Green Knight. F Madden. *Syre Gawayne*. A Collection of Ancient Romance-Poems. (1839). Ballantine Club. London.

[MENNER20] Robert J Menner. *Purity: A Middle English Poem*. (1888). Yale University Press. New Haven.

[PENNIGTON65] Ralph H Pennington. *Introductory Computer Methods and Numerical Analysis*. (1965). Collier-Macmillan. London. Chapter XII deals with the solution of systems of Linear Equations with worked examples. .

[SPEARING70] A C Spearing. *The Gawain-Poet*. A Critical Study. (1970). Cambridge University Press. London and New York.

[SPENDAL76] R J Spendal. *The Manuscript Capitals in "Cleanness"*. Notes and Queries. 221. 340-341. (1976).